THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 1

§2) 5) Let $z_k = x_k + iy_k$, where $k = 1, 2$. Then

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \\ &= x_2 x_1 - y_2 y_1 + i(x_2 y_1 + x_1 y_2) \\ &\quad \text{(by the commutativity of addition and multiplication of real numbers)} \\ &= (x_2 + iy_2)(x_1 + iy_1) \\ &= z_2 z_1. \end{aligned}$$

Hence multiplication of complex number is commutative.

§2) 6) (a) Let $z_k = x_k + iy_k$, where $k = 1, 2, 3$. Then

$$\begin{aligned} (z_1 + z_2) + z_3 &= [(x_1 + iy_1) + (x_2 + iy_2)] + (x_3 + iy_3) \\ &= ((x_1 + x_2) + i(y_1 + y_2)) + (x_3 + iy_3) \\ &= ((x_1 + x_2) + x_3) + i((y_1 + y_2) + y_3) \\ &= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3)) \\ &\quad \text{(by the associativity of addition of real numbers)} \\ &= (x_1 + iy_1) + ((x_2 + x_3) + i(y_2 + y_3)) \\ &= (x_1 + iy_1) + [(x_2 + iy_2) + (x_3 + iy_3)] \\ &= z_1 + (z_2 + z_3) \end{aligned}$$

Hence addition of complex number is associative.

(b) Let $z = x + iy$ and $z_k = x_k + iy_k$, where $k = 1, 2$. Then

$$\begin{aligned} z(z_1 + z_2) &= (x + iy)[(x_1 + iy_1) + (x_2 + iy_2)] \\ &= (x + iy)((x_1 + x_2) + i(y_1 + y_2)) \\ &= x(x_1 + x_2) - y(y_1 + y_2) + i(x(y_1 + y_2) + (x_1 + x_2)y) \\ &= [xx_1 - yy_1 + i(xy_1 + x_1 y)] + [xx_2 - yy_2 + i(xy_2 + x_2 y)] \\ &= zz_1 + zz_2 \end{aligned}$$

Hence the distributive law is true for complex number.

§3) 8) First of all, note that for any $m, k \in \mathbb{N}$ with $k \leq m$,

$$\begin{aligned}\binom{m}{k-1} + \binom{m}{k} &= \frac{m!}{(k-1)!(m-k+1)!} + \frac{m!}{k!(m-k)!} \\ &= \frac{m!(k+m-k+1)}{k!(m+1-k)!} \\ &= \frac{(m+1)!}{k!(m+1-k)!} \\ &= \binom{m+1}{k}\end{aligned}$$

Let $P(n)$ be the following proposition:

if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

When $n = 1$, $P(1)$ is clearly true.

Assume that $P(m)$ is ture fore some $m \in \mathbb{N}$, i.e.

if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^m = \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k}$$

When $n = m + 1$,

$$\begin{aligned}(z_1 + z_2)^{m+1} &= (z_1 + z_2)(z_1 + z_2)^m \\ &= (z_1 + z_2) \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k} \qquad \text{(by induction hypothesis)} \\ &= \sum_{k=0}^{m} \binom{m}{k} z_1^{k+1} z_2^{m-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k+1} \\ &= z_1^{m+1} + \sum_{k=0}^{m-1} \binom{m}{k} z_1^{k+1} z_2^{m-k} + \sum_{k=1}^{m} \binom{m}{k} z_1^k z_2^{m-k+1} + z_2^{m+1} \\ &= z_1^{m+1} + \sum_{k=1}^{m} \binom{m}{k-1} z_1^k z_2^{m-k+1} + \sum_{k=1}^{m} \binom{m}{k} z_1^k z_2^{m-k+1} + z_2^{m+1} \\ &= z_1^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k-1} + \binom{m}{k}\right] z_1^k z_2^{m-k+1} + z_2^{m+1} \\ &= z_1^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^k z_2^{m-k+1} + z_2^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m}{k} z_1^k z_2^{m+1-k}\end{aligned}$$

Hence $P(m + 1)$ is true. By M.I., $P(n)$ is true $\forall\ n \in \mathbb{N}$.

§5) 4) Let $z = x + iy$. Note that

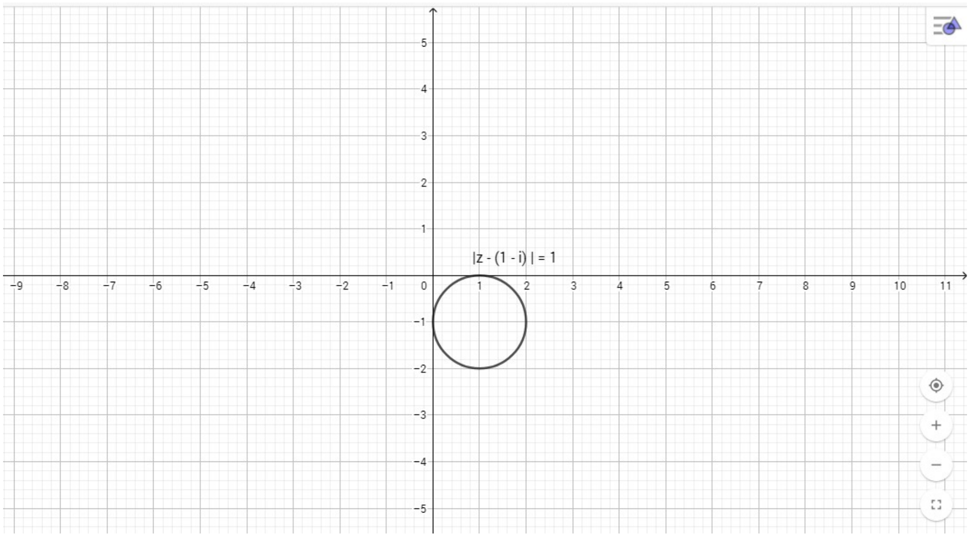
$$
\begin{aligned}
& (|x| - |y|)^2 \geq 0 \\
\implies & |x|^2 - 2|x||y| + |y|^2 \geq 0 \\
\implies & |x|^2 + |y|^2 \geq 2|x||y| \\
\implies & 2(|x|^2 + |y|^2) \geq |x|^2 + 2|x||y| + |y|^2 \\
\implies & 2(|z|) \geq (|\operatorname{Re}(z)| + |\operatorname{Im}(z)|)^2
\end{aligned}
$$

Since $2|z|$ and $|\operatorname{Re}(z)| + |\operatorname{Im}(z)|$ are both non-negative, we have

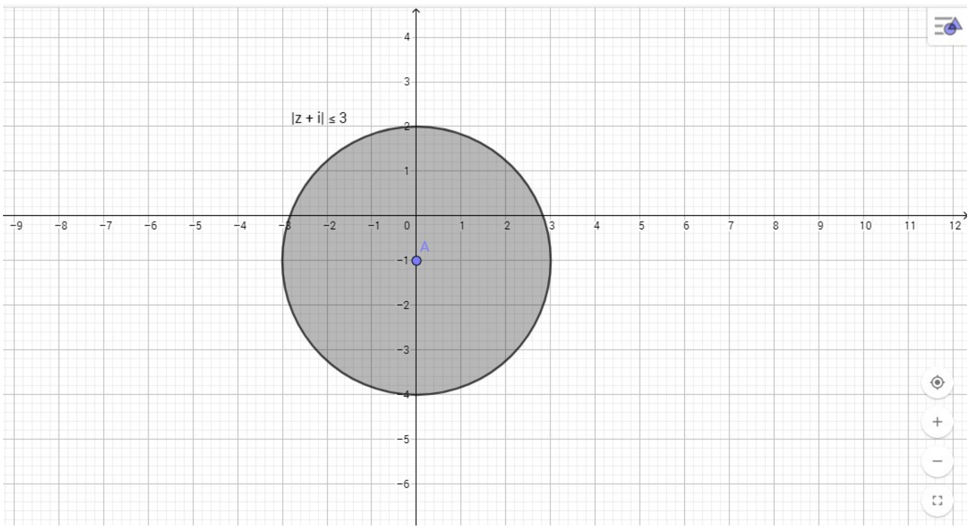
$$\sqrt{2}|z| \geq |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$$

§5) 5) The following pictures are drawn by Geogebra (https://www.geogebra.org/classic)

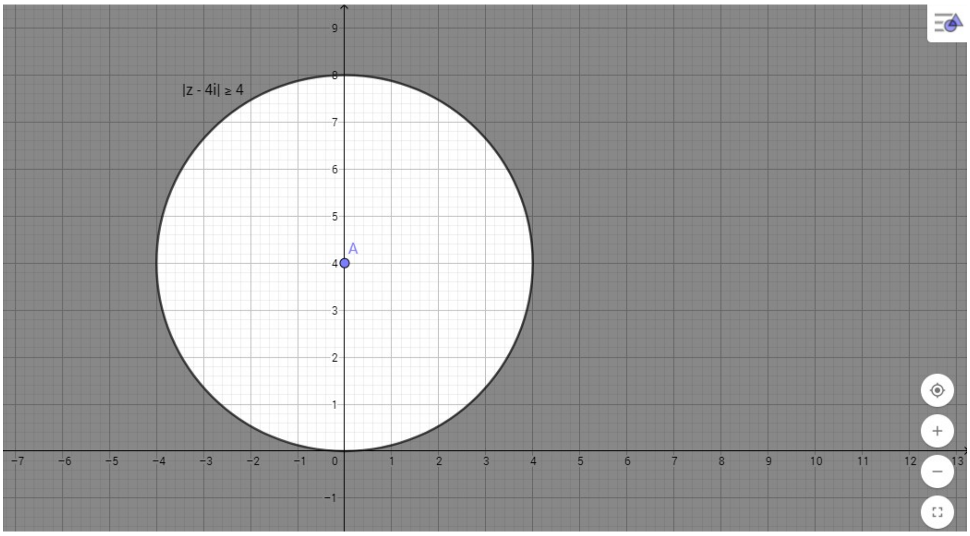
a)



b)



c)



§6) 9) Note that $z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$. By triangle inequality, for $|z| = 2$ we have

$$|z^2 - 1| \geq |z^2| - 1 = |z|^2 - 1 = (2)^2 - 1 = 3$$

and

$$|z^2 - 3| \geq |z^2| - 3 = |z|^2 - 3 = (2)^2 - 3 = 1.$$

Therefore,

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^2 - 1||z^2 - 3|} \leq \frac{1}{(3)(1)} = \frac{1}{3}.$$

§6) 15) a)

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\ &= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2} \end{aligned}$$

b) For any complex number $z = x + iy$, we have

$$z + \overline{z} = 2x = 2\,\mathrm{Re}(z) \tag{1}$$

$$x \leq |x| = \sqrt{|x|^2} \leq \sqrt{|x|^2 + |y|^2} = |z| \tag{2}$$

As a result,

$$\begin{aligned} z_1\overline{z_2} + \overline{z_1\overline{z_2}} &= 2\,\mathrm{Re}(z_1\overline{z_2}) && \text{(by (1))} \\ &\leq 2|z_1||\overline{z_2}| && \text{(by (2))} \\ &= 2|z_1||z_2| \end{aligned}$$

c) By (a) and (b), we have

$$\begin{aligned}|z_1+z_2|^2 &= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + z_2\overline{z_2} \\ &\le |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1|+|z_2|)^2.\end{aligned}$$

Since both $|z_1+z_2|$ and $|z_1|+|z_2|$ are non-negative, we have

$$|z_1+z_2| \le |z_1|+|z_2|.$$

§9) 9) Note that

$$\begin{aligned}(1+z+z^2+\cdots+z^n)(1-z) &= 1+z+z^2+\cdots+z^n - z - z^2 - z^3 - \cdots - z^{n+1} \\ &= 1 - z^{n+1}\end{aligned}$$

Since $z \neq 1$, we have

$$1+z+z^2+\cdots+z^n = \frac{1-z^{n+1}}{1-z}.$$

Put $z = e^{i\theta}$, we have

$$1+e^{i\theta}+e^{i2\theta}+\cdots+e^{in\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}. \qquad (\star)$$

Note that the real part of LHS is given by

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta.$$

On the other hand, the RHS is given by

$$\begin{aligned}\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} &= \frac{e^{i\frac{(n+1)\theta}{2}}}{e^{i\frac{\theta}{2}}} \frac{e^{i\frac{(n+1)\theta}{2}} - e^{-i\frac{(n+1)\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \\ &= e^{i\frac{n\theta}{2}} \frac{2i\sin(\frac{(n+1)\theta}{2})}{2i\sin(\frac{\theta}{2})}.\end{aligned}$$

Hence the real part of RHS is given by

$$\begin{aligned}\frac{\cos\frac{n\theta}{2}\sin(\frac{(n+1)\theta}{2})}{\sin(\frac{\theta}{2})} &= \frac{\sin(\frac{(n+1)\theta}{2}+\frac{n\theta}{2}) + \sin(\frac{(n+1)\theta}{2}-\frac{n\theta}{2})}{2\sin(\frac{\theta}{2})} \\ &= \frac{1}{2} + \frac{\sin(\frac{(2n+1)\theta}{2})}{\sin(\frac{\theta}{2})}.\end{aligned}$$

By comparing the real part of $(\star)$, we get the Lagrange's trigonometric identity.

§9) 10) Note that

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= e^{i3\theta} \\ &= (e^{i\theta})^3 \\ &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

a) By comparing the real part of the above equation, we have

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta.$$

b) By comparing the imaginary part of the above equation, we have

$$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta.$$